

# On scattering of small energy solutions of non autonomous hamiltonian nonlinear Schrödinger equations

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## Abstract

We revisit a result by Cuccagna, Kirr and Pelinovsky about the cubic nonlinear Schrödinger equation (NLS) with an attractive localized potential and a time-dependent factor in the nonlinearity. We show that, under generic hypotheses on the linearization at 0 of the equation, small energy solutions are asymptotically free. This is yet a new application of the hamiltonian structure, continuing a program initiated in a paper by Bambusi and Cuccagna.

## 1 Introduction

We consider for  $(t, x) \in \mathbb{R} \times \mathbb{R}^3$  the nonlinear Schrödinger equation (1.1)

$$iu_t(t, x) = \mathcal{H}u(t, x) + \gamma(t)|u(t, x)|^2u(t, x), \quad u(0, x) = u_0(x). \quad (1.1)$$

Here  $\mathcal{H} := -\Delta + V(x) + \underline{c}$  with  $\underline{c} > 0$  a constant.  $\gamma(t)$  is of form

$$\gamma(t) = \gamma_0 + \gamma_1 \cos(t), \quad \gamma_0, \gamma_1 \in \mathbb{R}, \gamma_1 \neq 0. \quad (1.2)$$

We will assume the following hypotheses.

- (H1)  $V(x)$  is a real valued Schwartz function.
- (H2) We assume  $\mathcal{H} \geq 0$ .
- (H3) The set of eigenvalues  $\sigma_d(\mathcal{H})$  is contained in  $[0, \underline{c}]$ . Specifically, we assume that  $0 \in \sigma_d(\mathcal{H})$  and that the sum of the multiplicities of the eigenvalues is  $n + 1$ . We write  $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n$  repeating each eigenvalue a number of times equal to its multiplicity (notice that it is well known that (H1)–(H2) implies  $\dim \ker \mathcal{H} \leq 1$ ).
- (H4)  $\underline{c}$  is not an eigenvalue or a resonance for  $\mathcal{H}$ , i.e. there are no nonzero solutions of  $\Delta u = Vu$  in  $\mathbb{R}^3$  with  $|u(x)| \lesssim \langle x \rangle^{-1}$ .

(H5)  $\underline{c} \notin \mathbb{N}$ .

(H6)  $\forall j = 1, \dots, n$  there exists  $N_j \in \mathbb{N}$  such that  $N_j \lambda_j < \underline{c} < (N_j + 1) \lambda_j$ . Notice that  $N_1 = \sup_j N_j$ .

Let now  $[\underline{c}] \in \mathbb{Z}$  be the integral part of  $\underline{c}$ , defined by  $[\underline{c}] \leq \underline{c} < [\underline{c}] + 1$  and set  $N = \max\{N_1, [\underline{c}]\}$ .

(H7) For any multi index  $\mu \in \mathbb{Z}^{n+1}$  with  $|\mu| := |\mu_0| + \dots + |\mu_k| \leq 2N + 1$  and any  $m \in \mathbb{Z}$  with  $|m| \leq N$ , we have  $\mu \cdot \lambda + m \neq \underline{c}$ .

(H8) If  $0 < \lambda_{j_1} < \dots < \lambda_{j_k}$  are  $k$  distinct  $\lambda$ 's,  $\mu \in \mathbb{Z}^k$  satisfies  $|\mu| \leq 4N + 2$  and  $m \in \mathbb{Z}$  satisfies  $|m| \leq 2N$ , then we have

$$\mu_1 \lambda_{j_1} + \dots + \mu_k \lambda_{j_k} + m = 0 \iff \mu = 0 \text{ and } m = 0.$$

(H9) The Fermi golden rule Hypothesis (H9') in subsection 4.1, see (4.29), holds.

(H10) We have  $\gamma_1 \neq 0$  in (1.2).

**Theorem 1.1.** *Let  $u(t, x)$  be a solution to (1.1). Assume (H1)–(H10). Then, there exist an  $\epsilon_0 > 0$  and a  $C > 0$  such that if  $\|u_0\|_{H^1} < \epsilon$  with  $\epsilon \in (0, \epsilon_0)$ , there exist  $h_{\pm} \in H^1$  with  $\|h_{\pm}\|_{H^1} \leq C\|u_0\|_{H^1}$  such that*

$$\lim_{t \rightarrow \pm\infty} \|u(t, \cdot) - e^{it\Delta} h_{\pm}\|_{H^1} = 0. \quad (1.3)$$

*It is possible to write  $u(t, x) = A(t, x) + \tilde{u}(t, x)$  with  $|A(t, x)| \leq C_N(t) \langle x \rangle^{-N}$  for any  $N$ , with  $\lim_{|t| \rightarrow \infty} C_N(t) = 0$  and such that for any pair  $(r, p)$  which is admissible, by which we mean that*

$$2/r + 3/p = 3/2, \quad 6 \geq p \geq 2, \quad r \geq 2, \quad (1.4)$$

*we have*

$$\|\tilde{u}\|_{L_t^r(\mathbb{R}, W_x^{1,p})} \leq C\|u_0\|_{H^1}. \quad (1.5)$$

*Remark 1.2.* When  $\gamma_1 = 0$  equation (1.1) admits standing waves of arbitrarily small energy, by simple bifurcation theory. So (H10), that is  $\gamma_1 \neq 0$ , is an essential hypothesis.

*Remark 1.3.* Theorem 1.1 is a generalization of the main result of [CKP] which focuses on the special case  $n = 0$  and  $\underline{c} < 1$ . In the special case treated in [CKP], our proof is particularly simple (although this is here obscured by our emphasis on easing the restrictions on  $\sigma(\mathcal{H})$  of [CKP]). Notice that we do not obtain analogues of the decay formulas (1.12)–(1.14) [CKP] because our initial data are not required to satisfy  $\int_{\mathbb{R}^3} |x|^\sigma |u_0(x)|^2 dx \ll 1$  for a  $\sigma > 5$ , like in [CKP]: if one merely asks  $\|u_0\|_{H^1} \ll 1$ , as we do, the decay formulas (1.12)–(1.14) [CKP] are not true.

*Remark 1.4.* At the beginning of section 7 [CKP] is mentioned, without details, the possibility of proving the main result of [CKP] in the case  $n = 0$  and  $\underline{c} > 1$ . In fact this case is substantially harder, even more if also  $n \geq 1$ . Treating these cases is what we accomplish here.

*Remark 1.5.* We treat a larger class of solutions than [CKP] and we are able to draw stronger conclusions. For instance, in [CKP] the only energy bound proved is of the form  $\|u(t)\|_{H^1} \lesssim \epsilon \log(\epsilon^4 t)$ : here we prove  $\|u(t)\|_{H^1} \lesssim \epsilon$ .

*Remark 1.6.* We choose the nonlinearity  $|u|^2 u$  to simplify exposition. Indeed in this case the energy  $E(t, u)$ , see (2.2), is smooth in  $(t, u) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ , and Theorem 3.6 below is easier to prove. But, with more effort and with essentially the same argument, we could have considered a nonlinearity  $\beta(|u|^2)u$ , with  $\beta(0) = 0$ ,  $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$  and s.t. there exists a  $p \in (1, 5)$  s.t. for every  $k \geq 0$  there is a fixed  $C_k$  with  $\left| \frac{d^k}{dv^k} \beta(v^2) \right| \leq C_k |v|^{p-k-1}$  if  $|v| \geq 1$ , see [BC, C1].

*Remark 1.7.* We choose space dimension 3 only for definiteness. It is possible to prove a similar theorem for any spatial dimension. In low dimensions, 1 and 2, there are no endpoint Strichartz inequalities, but [M1, M2] gives us good surrogates. Notice that the proofs in [M1, M2] can be substantially simplified, following the ideas from Lemma 3.2 to Lemma 3.6 in [CT]. The nonlinearity  $|u|^2 u$  can be treated in space dimension 2. For space dimension 1, given our need of Strichartz estimates to close nonlinear estimates, it is necessary to work with  $\beta(|u|^2)u$ ,  $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$  with  $\beta(0) = \beta'(0) = 0$ . Hence, and this is an important technical constraint, the nonlinearity is 0 at least at fifth order at  $u = 0$ . For 1 D the nonlinearity  $|u|^2 u$  is difficult, being *long range*, so it remains an open problem. Notice that for space dimension 1, the energy  $E(t, u)$  is smooth in  $(t, u) \in \mathbb{R} \times H^1(\mathbb{R}^3)$  for all  $\beta(|u|^2)u$  with  $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ , so Theorem 3.6 below can be easily proved with  $\beta(|u|^2)u$ . Summing up, the second of the open problems stated at the end of p.115 [CKP] is mostly solvable, but is unsolved in the case of the nonlinearity  $|u|^2 u$  in 1 D.

*Remark 1.8.* Hypothesis (H9) probably holds for generic  $V$ . Under the hypotheses on  $\sigma(\mathcal{H})$  of [CKP], hypothesis (H9) is the same of formula (1.9) in [CKP]. It is easy to show that (1.9) in [CKP] holds for generic  $V$ , if additionally  $|V(x)| \leq C e^{-a|x|}$  for  $a > 0$ , see the proof of Proposition 2.2 and Remark A.1 in [BC]. Notice that Proposition 2.2 [BC] proves that the analogue of (H9) in [BC] is true for generic  $\beta(|u|^2)$  with fixed  $V$  (with  $V$  exponentially decreasing and with simple eigenvalues): probably an analogous proof yields (H9) for generic pairs  $(\beta(|u|^2), V)$ .

*Remark 1.9.* The function  $\gamma(t)$  in (1.2) is particularly simple. This simplifies the exposition. But similar arguments work if  $\gamma(t)$  is a higher degree trigonometric polynomial, or if, for  $P(x_1, y_1, \dots, x_A, y_A)$  a real valued nonconstant polynomial,  $\gamma(t) = P(\cos(\omega_1 t), \sin(\omega_1 t), \dots, \cos(\omega_A t), \sin(\omega_A t))$ , adding appropriate non resonance hypotheses on these frequencies, the eigenvalues of  $\mathcal{H}$  and  $\underline{c}$ .

We recall that [CKP] shows that (under very restrictive hypotheses) nonlinear coupling of continuous and discrete modes is responsible of leaking of energy

from discrete modes into radiation, where linear dispersion occurs. This is analogous in linear theory to the Stark effect, see [Y], and to effects of disturbances on ground states, see [KW] and references therein. Nonlinear coupling of continuous and discrete modes is exploited in [SW, BC] for a proof of scattering of small energy solutions of the nonlinear Klein Gordon equation (NLKG) with discrete modes. The same idea is exploited in a substantial number of papers dealing with asymptotic stability of ground states of the nonlinear Schrödinger equation (NLS), see [C1] and therein for more references. From the beginning, at least 15 years ago [S, BP, SW], it was clear how coupling should act. See also the improvements on [BP, SW] contained in [CM, GS]. In particular, attention was drawn to the sign of specific coefficients of the discrete mode equations. This sign is responsible for friction on the discrete modes. Since the system is hamiltonian, the energy is conserved and simply is moving from discrete to continuous modes. Except for special cases though, it was not clear how to prove the sign. Emphasis was rightly attached to the necessity of spectral resonance between eigenvalues and continuous spectrum of the linearization. Indeed when this resonance is absent, like for the examples of discrete NLS in [C2], the discrete modes persist. But for continuous NLS there is always spectral resonance. What was not well appreciated was the crucial role of the hamiltonian structure. Attempts to prove friction without exploiting the hamiltonian structure were extremely complex, see [Gz], or somewhat indirect and unsatisfying, see [CM], and confined to the case of just 1 discrete mode. In the case of multiple discrete modes, friction was proved only in special cases with the eigenvalues close to the continuous spectrum, [T, C3, GW]. The first reference which recognizes the relevance of the hamiltonian structure seems to be [C3]. In [BC, C1] we were finally able to exploit the intuition of [C3] and to exploit the hamiltonian structure to prove the friction on the discrete modes under very general hypotheses on the spectrum. By applying a conceptually simple form of the Birkhoff normal form argument, [BC] is able to extend the result in [SW] by dropping the spectral assumptions in [SW]. Along the same lines, [C1] proves asymptotic stability of ground states of the NLS. The situation in [C1] is harder, since rather than an equilibrium point there is an invariant manifold. In the present paper we return to the easier setting of [BC] where the issue is to prove the asymptotic stability of the equilibrium point 0. So, we improve [CKP] in the same way [BC] improves [SW]. As in [BC], the Birkhoff normal form helps us to prepare the system. We then conclude the proof in Section 4 with somewhat standard arguments, derived most directly from [CM, BC, C1], but which are a simplification and generalization of arguments already in [BP, SW]. Notice that these arguments are simpler than [CKP]. For example, there is no need of hierarchies of Banach spaces like in [CKP]. Finally, for the physical relevance, some interesting open problems and more context and references, we refer to [CKP].

We end the introduction with some notation. Given two functions  $f, g : \mathbb{R}^3 \rightarrow \mathbb{C}$  we set  $\langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$ . For any  $k, s \in \mathbb{R}$  we set

$$H^{k,s}(\mathbb{R}^3) = \{f : \mathbb{R}^3 \rightarrow \mathbb{C} \text{ s.t. } \|f\|_{H^{s,k}} := \|\langle x \rangle^s (-\Delta + 1)^k f\|_{L^2} < \infty\}.$$

We set  $\mathcal{S}(\mathbb{R}^3) = \cap_{s,k=0}^{\infty} H^{k,s}(\mathbb{R}^3)$ . We set  $L^{2,s} = H^{0,s}$ ,  $L^2 = L^{2,0}$ ,  $H^k = H^{2,0}$ . Sometimes, to emphasize that these spaces refer to spatial variables, we will denote them by  $W_x^{k,p}$ ,  $L_x^p$ ,  $H_x^k$ ,  $H_x^{k,s}$  and  $L_x^{2,s}$ . For  $I$  an interval and  $Y_x$  any of these spaces, we will consider Banach spaces  $L_t^p(I, Y_x)$  with mixed norm  $\|f\|_{L_t^p(I, Y_x)} := \|\|f\|_{Y_x}\|_{L_t^p(I)}$ . Given an operator  $A$ , we will denote by  $R_A(z) = (A - z)^{-1}$  its resolvent. We set  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

## 2 Local well posedness and Hamiltonian structure

The first step is the locally well-posed in  $H^1(\mathbb{R}^3)$  of the initial-value problem (1.1), see Theorem 2 [CKP]:

**Theorem 2.1.** *For every  $u_0 \in H^1(\mathbb{R}^3)$  there exists a unique solution  $u(t)$  of the initial-value problem (1.1) defined on a maximal interval  $t \in [0, T_{max})$  such that*

$$u \in C^1([0, T_{max}), H^{-1}) \cap C([0, T_{max}), H^1), \quad (2.1)$$

where if  $T_{max} < \infty$  then  $\lim_{t \rightarrow T_{max}} \|u(t)\|_{H^1} = \infty$ . Moreover,  $\|u(t)\|_2 \equiv \|u(0)\|_2$ ,  $\forall t \in [0, T_{max})$ , and  $u(t)$  depends continuously on the initial data, i.e. if  $\lim_{n \rightarrow \infty} u_0^n = u_0$  in  $H^1(\mathbb{R}^3)$  then for any closed interval  $I \subset [0, T_{max})$  the solution  $u^n(t)$  of the problem (1.1) with initial data  $u_0^n$  is defined on  $I$  for sufficiently large  $n$ , and  $\lim_{n \rightarrow \infty} u^n(t) = u(t)$  in  $C(I, H^1)$ .

The next step is about the hamiltonian nature of (1.1), which is neglected in [CKP] but which in fact is crucial. So we spend the rest of the section to discuss the hamiltonian set up. We have an energy functional

$$\begin{aligned} E(t, u) &= E_K(u) + E_P(t, u) \\ E_K(u) &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla \bar{u} dx + \underline{c} \int_{\mathbb{R}^3} u \bar{u} dx + \int_{\mathbb{R}^3} V u \bar{u} dx \\ E_P(t, u) &= \gamma(t) \int_{\mathbb{R}^3} \frac{|u|^4}{4} dx \end{aligned} \quad (2.2)$$

Equation (1.1) can be written as

$$i\dot{u} = \partial_{\bar{u}} E(t, u). \quad (2.3)$$

We consider eigenfunctions  $\phi_j(x)$  with eigenvalue  $\lambda_j$ :  $\mathcal{H}\phi_j = \lambda_j\phi_j$ . They can be normalized so that they are real valued and  $\langle \phi_j, \phi_\ell \rangle = \delta_{j\ell}$ . The  $\phi_j(x)$  are smooth and satisfy for some fixed  $a > 0$  and all multi indexes  $\alpha$

$$\sup_{x \in \mathbb{R}^3, j=0,n} e^{a|x|} |\partial_x^\alpha \phi_j(x)| < \infty. \quad (2.4)$$

We have the  $\mathcal{H}$  decomposition

$$L^2(\mathbb{R}^3, \mathbb{C}) = \ker(\mathcal{H}) \oplus_{j=1}^n \ker(\mathcal{H} - \lambda_j) \oplus L_c^2(\mathcal{H}). \quad (2.5)$$

Correspondingly we set,

$$u = z \cdot \phi + f, \text{ for } z \cdot \phi = \sum_{j=0}^n z_j \phi_j(x). \quad (2.6)$$

In  $L^2(\mathbb{R}^3, \mathbb{C}^2)$  we consider the symplectic form

$$\Omega_0((u, \bar{u}), (v, \bar{v})) := \langle u, \bar{v} \rangle - \langle \bar{u}, v \rangle \quad (2.7)$$

The proof of the following lemma is elementary:

**Lemma 2.2.** *Let  $P_c$  be the projection in  $L_c^2(\mathcal{H})$ . Consider the functions  $z_j = \langle u, \phi_j \rangle$ ,  $\bar{z}_j = \langle \bar{u}, \phi_j \rangle$ ,  $f = P_c u$  and  $\bar{f} = P_c \bar{u}$ . Then in terms of these functions we have*

$$\Omega_0 = \sum_{j=0}^n dz_j \wedge d\bar{z}_j + \langle f', \bar{f}' \rangle - \langle \bar{f}', f' \rangle, \quad (2.8)$$

where  $f'v = P_c v$  and  $\bar{f}'v = \overline{P_c v}$  and where  $\langle f', \bar{f}' \rangle$  (resp.  $\langle \bar{f}', f' \rangle$ ) acts on a pair  $(v_1, v_2)$  as  $\langle f'v_1, \bar{f}'v_2 \rangle$  (resp.  $\langle \bar{f}'v_1, f'v_2 \rangle$ ).

We consider now two additional variables  $(t, \tau) \in \mathbb{R}^2$  and we set

$$\Omega = \Omega_0 + i dt \wedge d\tau. \quad (2.9)$$

For a function  $F$  we call hamiltonian vector field  $X_F$  with respect to  $\Omega$  the field defined by  $\Omega(X_F, Y) = -i dF(Y)$ . For any vector  $\Sigma$  we set

$$\Sigma = \Sigma_t \frac{\partial}{\partial t} + \Sigma_\tau \frac{\partial}{\partial \tau} + \sum \Sigma_j \frac{\partial}{\partial z_j} + \sum \Sigma_{\bar{j}} \frac{\partial}{\partial \bar{z}_j} + \Sigma_f + \Sigma_{\bar{f}} \quad (2.10)$$

for

$$\begin{aligned} \Sigma_t &= dt(\Sigma), \quad \Sigma_\tau = d\tau(\Sigma), \quad \Sigma_j = dz_j(\Sigma) \\ \Sigma_{\bar{j}} &= d\bar{z}_j(\Sigma), \quad \Sigma_f = f'\Sigma, \quad \Sigma_{\bar{f}} = \bar{f}'\Sigma. \end{aligned} \quad (2.11)$$

A differential 1-form  $\alpha$  decomposes as

$$\alpha = \alpha^t dt + \alpha^\tau d\tau + \sum \alpha^j dz_j + \sum \alpha^{\bar{j}} d\bar{z}_j + \langle \alpha^f, f' \rangle + \langle \alpha^{\bar{f}}, \bar{f}' \rangle, \quad (2.12)$$

where  $\langle \alpha^f, f' \rangle$  (resp.  $\langle \alpha^{\bar{f}}, \bar{f}' \rangle$ ) acts on a vector  $v$  as  $\langle \alpha^f, f'v \rangle$  (resp. as  $\langle \alpha^{\bar{f}}, \bar{f}'v \rangle$ ). Then

$$\begin{aligned} (X_F)_t &= -\frac{\partial F}{\partial \tau}, \quad (X_F)_\tau = \frac{\partial F}{\partial t}, \quad (X_F)_j = -i \frac{\partial F}{\partial \bar{z}_j} \\ (X_F)_{\bar{j}} &= i \frac{\partial F}{\partial z_j}, \quad (X_F)_f = -i \nabla_{\bar{f}} F, \quad (X_F)_{\bar{f}} = i \nabla_f F. \end{aligned} \quad (2.13)$$

We call Poisson bracket of two functions with respect to  $\Omega$  the function

$$\begin{aligned} \{F, G\} &:= dF(X_G) = \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial G}{\partial \tau} + \\ &+ i \sum_{j=1}^n \left( \frac{\partial F}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j} - \frac{\partial F}{\partial z_j} \frac{\partial G}{\partial \bar{z}_j} \right) + i \langle \nabla_{\bar{f}} F, \nabla_f G \rangle - i \langle \nabla_{\bar{f}} G, \nabla_f F \rangle. \end{aligned} \quad (2.14)$$

We consider the new hamiltonian

$$H := H_F + E_P(t, u), \quad H_F := E_K - \tau. \quad (2.15)$$

Then the corresponding hamiltonian system is

$$\begin{aligned} \frac{\partial}{\partial s} z_j &= -i \frac{\partial H}{\partial \bar{z}_j}, \quad \frac{\partial}{\partial s} f = -i \nabla_{\bar{f}} H \\ \frac{d}{ds} t &= -\frac{\partial}{\partial \tau} H = 1, \quad \frac{d}{ds} \tau = \frac{\partial}{\partial t} H. \end{aligned} \quad (2.16)$$

It is easy to see that (2.16) and (2.3) are equivalent. Plugging the decomposition (2.6) in the energy (2.2), it is easy to see the following equality:

$$H_F = \sum_{j=1}^n \lambda_j z_j \bar{z}_j + \langle \mathcal{H} f, \bar{f} \rangle - \tau. \quad (2.17)$$

Once the natural coordinates (2.6) are introduced, [CKP] starts a normal form argument to simplify the system. Here we do the same, but we want to preserve the hamiltonian structure. So we need canonical transformations, which is the theme of section 3.

## 3 Canonical transformations

### 3.1 Lie transform

For  $m_0 \in \mathbb{N}_0$ ,  $M_0 \in \mathbb{N}$  we consider functions

$$\begin{aligned} \chi &= \sum_{\ell=-m_0}^{m_0} e^{i\ell t} \left[ \sum_{|\mu|=|\nu|=M_0+1} a_{\ell\mu\nu} z^\mu \bar{z}^\nu + \sum_{\substack{|\mu|=M_0 \\ |\mu|=|\nu|-1}} z^\mu \bar{z}^\nu \langle \Phi_{\ell\mu\nu}, f \rangle \right. \\ &\quad \left. + \sum_{\substack{|\nu|=M_0 \\ |\mu|=|\nu|+1}} z^\mu \bar{z}^\nu \langle \Psi_{\ell\mu\nu}, \bar{f} \rangle \right]. \end{aligned} \quad (3.1)$$

We assume

$$\bar{a}_{\ell\mu\nu} = a_{-\ell\nu\mu}, \quad \bar{\Phi}_{\ell\mu\nu}(x) = \Psi_{-\ell\nu\mu}(x). \quad (3.2)$$

We assume  $\Phi_{m\mu\nu}(x) \in \mathcal{S}(\mathbb{R}^3)$ . Denote by  $\mathfrak{F}^s$  the flow of the Hamiltonian vector field  $X_\chi$ . The *Lie transform*  $\mathfrak{F} = \mathfrak{F}^s|_{s=1}$  is defined in a sufficiently small neighborhood of the origin and is a canonical transformation. Then we have, with  $(z', f') = \phi(t, z, f)$ ,

$$(\tau', t', z', f') := \mathfrak{F}(\tau, t, z, f) = (\tau + \psi(t, z, f), t + 1, \phi(t, z, f)). \quad (3.3)$$

**Lemma 3.1.** *Consider the  $\chi$  in (3.1) and its Lie transform  $\mathfrak{F}$ . Fix any  $n_0 \in \mathbb{N} \cup \{0\}$ . Then there are  $\mathcal{G}(t, z, f)$  and  $\Gamma_j(t, z, f)$  such that for any pair  $(-K', -S')$  we have the following properties.*

- (1)  $\Gamma_j \in C^\infty(\mathcal{U}^{-K', -S'}, \mathbb{R})$ , with  $\mathcal{U}^{-K', -S'} = \tilde{\mathcal{U}}^{-K', -S'} \times \mathbb{R}$ , with  $\tilde{\mathcal{U}}^{-K', -S'} \subset \mathbb{C}^{n+1} \times H_c^{-K', -S'}$  an appropriately small neighborhood of the origin.
- (2)  $\mathcal{G} \in C^\infty(\mathcal{U}^{-K', -S'}, H_c^{K, S})$  for any  $K, S$ .
- (3) The transformation  $\phi(t, z, f)$  is such that we have

$$\begin{aligned} z'_j &= z_j + U_j + \Gamma_j(t, z, f), \\ f' &= f + V + \mathcal{G}(t, z, f) \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} U_j &= \sum_{l=1}^{n_0} \frac{1}{l!} \mathcal{L}_\chi^l(z_j), \quad V = \sum_{l=1}^{n_0} \frac{1}{l!} \mathcal{L}_\chi^l(f), \quad \mathcal{L}_\chi^l(g) := \{\underbrace{\dots\{g, \chi\}\dots\chi}_l\} \\ \{f, \chi\} &:= -i \nabla_{\bar{f}} \chi = -i \sum_{\ell=-m_0}^{m_0} e^{i\ell t} \sum_{\substack{|\nu|=M_0 \\ |\mu|=|\nu|+1}} z^\mu \bar{z}^\nu \Psi_{\ell\mu\nu}(x), \\ \mathcal{L}_\chi^l(f) &:= -i \sum_{\ell=-\ell_0}^{m_0} e^{i\ell t} \sum_{\substack{|\nu|=M_0 \\ |\mu|=|\nu|+1}} \mathcal{L}_\chi^{l-1}(z^\mu \bar{z}^\nu) \Psi_{\ell\mu\nu}(x). \end{aligned} \quad (3.5)$$

- (4)  $\mathcal{L}_\chi^l(z_j)$  is a homogeneous polynomial of degree  $2lM_0 + 1$  in  $(z, \bar{z})$ ,  $\langle \Phi_{\ell\mu\nu}, f \rangle$  and  $\langle \Psi_{\ell\mu\nu}, \bar{f} \rangle$ . Same statement is true for  $\mathcal{L}_\chi^l(f)$ .
- (5) There exists a constant  $C = C(K, S, K', S')$  such that near 0

$$|\Gamma_j| + \|\mathcal{G}\|_{H^{K, S}} \leq C(|z| + \|f\|_{H^{-K', -S'}})^{2(n_0+1)M_0+1}. \quad (3.6)$$

*Proof.* Recall that for any function  $\psi$ , we have  $\frac{d}{ds}(\psi \circ \mathfrak{F}^s) = \{\psi, \chi\} \circ \mathfrak{F}^s$ . Then (3.4) follow by Taylor expansion with reminders

$$\begin{aligned} \Gamma_j &:= \int_0^1 \frac{(1-s)^{n_0}}{n_0!} \{\underbrace{\dots\{z_j, \chi\}\dots\chi}_{n_0+1}\} \circ \mathfrak{F}_s ds \\ \mathcal{G} &:= \int_0^1 \frac{(1-s)^{n_0}}{n_0!} \{\underbrace{\dots\{f, \chi\}\dots\chi}_{n_0+1}\} \circ \mathfrak{F}_s ds. \end{aligned} \quad (3.7)$$



(3.6) follows closing up inequalities with a standard argument, see proof of Lemma 4.3 [BC]. Claim (4) is true for  $l = 1$ , see for example  $\mathcal{L}_\chi(f)$  in (3.5). The general case follows by induction.  $\square$

**Lemma 3.2.** *Consider*

$$g = e^{itm} z^\mu \bar{z}^\nu \langle \Phi, f \rangle^\alpha \langle \Psi, \bar{f} \rangle^\beta \langle f^a \bar{f}^b, \Psi_{ab} \rangle, \quad (3.8)$$

where for a multiindex  $\alpha = (\alpha_1, \dots, \alpha_M)$  we have set  $\langle \Phi, f \rangle^\alpha = \prod_{j=1}^M \langle \Phi_j, f \rangle^{\alpha_j}$  for  $\Phi_j \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$  and where  $\langle \Psi, \bar{f} \rangle^\beta$  has a similar meaning, i.e.  $\langle \Psi, \bar{f} \rangle^\beta = \prod_{j=1}^{\widetilde{M}} \langle \Psi_j, \bar{f} \rangle^{\beta_j}$  for  $\Psi_j \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$  for a multiindex  $\beta = (\beta_1, \dots, \beta_{\widetilde{M}})$ . Assume:

- (i)  $a + b \leq 4$ ;
- (ii)  $|\mu| + |\alpha| + a = |\nu| + |\beta| + b = L + 1$ ;
- (iii)  $\Psi_{ab} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$  for  $a + b < 4$ ; if  $a + b = 4$  we assume  $\Psi_{ab} = \frac{1}{4} \delta_{ab}$  and  $L = 2$ ;
- (iv)  $|m| \leq L$ .

Consider  $\chi$  as in (3.1). Then  $\mathcal{L}_\chi(g)$  is a finite sum of terms of the form

$$e^{itm'} z^{\mu'} \bar{z}^{\nu'} \langle \Phi', f \rangle^{\alpha'} \langle \Psi', \bar{f} \rangle^{\beta'} \langle f^{a'} \bar{f}^{b'}, \Psi'_{a'b'} \rangle \quad (3.9)$$

with:

- (1)  $a' + b' < 4$  and  $\Psi'_{a'b'} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ ;
- (2)  $|\mu'| + |\alpha'| + a' = |\nu'| + |\beta'| + b' = L' + 1$ , with  $L' = L + M_0$ ;
- (3)  $|m'| \leq m_0 + |m|$ .

In (3.9) the factors  $\langle \Phi', f \rangle^{\alpha'}$  and  $\langle \Psi', \bar{f} \rangle^{\beta'}$  are defined like the corresponding ones in (3.8) and involve functions  $\Phi'_j, \Psi'_j \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ .

*Proof.*  $\mathcal{L}_\chi(g)$  is obtained applying the Leibnitz rule to the rhs(3.8). Trivially, if  $a + b < 4$  then also  $a' + b' < 4$ . On the other hand, for  $a + b = 4$  we have  $g = \frac{1}{4} \|f\|_4^4$  by (iii). Then  $\mathcal{L}_\chi(g)$  is a linear combination of terms  $\langle \mathcal{L}_\chi(f) f \bar{f}^2, 1 \rangle$  and  $\langle \mathcal{L}_\chi(\bar{f}) \bar{f} f^2, 1 \rangle$ . If we apply the formula for  $\mathcal{L}_\chi(f)$  in (3.5) and by the fact that  $\mathcal{L}_\chi(\bar{f}) = \overline{\mathcal{L}_\chi(f)}$ , consequence of (3.2), we obtain claim (1). (3) follows by the fact that in (3.9) we have  $m' = \ell + m$  with  $|\ell| \leq m_0$  and  $m$  the same of (3.8). Finally, (2) is an elementary consequence of the Leibnitz rule and the definition of  $\chi$  in (3.1). For example, one of the terms in the expansion of  $\mathcal{L}_\chi(g)$  is

$$a e^{itm} z^\mu \bar{z}^\nu \langle \Phi, f \rangle^\alpha \langle \Psi, \bar{f} \rangle^\beta \langle \mathcal{L}_\chi(f) f^{a-1} \bar{f}^b, \Psi_{ab} \rangle.$$

If we substitute the formula for  $\mathcal{L}_\chi(f)$  in (3.5) we get a linear combination of terms

$$e^{it(m+\ell)} z^{\mu+\mu''} \bar{z}^{\nu+\nu''} \langle \Phi, f \rangle^\alpha \langle \Psi, \bar{f} \rangle^\beta \langle f^{a-1} \bar{f}^b, \Psi_{\ell\mu''\nu''} \Psi_{ab} \rangle,$$

where  $|\nu''| = M_0$ ,  $|\mu''| = M_0 + 1$  and  $\Psi_{\ell\mu''\nu''} \Psi_{ab} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ . This is of form (3.9) with

$$\begin{aligned} m' &= m + \ell, & \mu' &= \mu + \mu'', & \nu' &= \nu + \nu'', & \alpha' &= \alpha, \\ \beta' &= \beta, & a' &= a - 1, & b' &= b, \\ |\mu'| + |\alpha'| + a' &= |\mu| + |\alpha| + a - 1 + |\mu''| = L + M_0 + 1, \\ |\nu'| + |\beta'| + b' &= |\nu| + |\beta| + b + |\nu''| = L + M_0 + 1. \end{aligned}$$

□

**Lemma 3.3.** *Consider  $g$  like in (3.8), with the above conventions and with assumptions (i)–(iv) of Lemma 3.3.*

(a) *We have an expansion*

$$\begin{aligned} g \circ \mathfrak{F}_1 &= g + \mathcal{V} + \mathbf{G}, \quad \mathcal{V} = \sum_{l=1}^{n_0} \frac{1}{l!} \mathcal{L}_\chi^l(g), \\ |\mathbf{G}| &\leq C(|z| + \|f\|_{H^{-K', -S'}})^{2M_0(n_0+1)+2L-a-b} (|z| + \|f\|_{H^1})^{a+b}. \end{aligned} \quad (3.10)$$

(b) *Each  $\mathcal{L}_\chi^l(g)$  is a finite sum of terms of the form (3.9) with  $|m'| \leq lm_0 + |m|$ , where  $L' + 1 = |\mu'| + |\alpha'| + a' = |\nu'| + |\beta'| + b'$  satisfies  $L' = L + lM_0$ . We have  $a' + b' < 4$  and  $\Psi'_{a'b'} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ .*

*Proof.* (b) is obtained iterating the result in Lemma 3.3. We turn to the proof of (a). By Lemma 3.4,  $f \circ \mathfrak{F}_s - f \in C^\infty(\mathcal{U}^{-K', -S'}, H_c^{K, S})$ . Hence  $g \circ \mathfrak{F}_s$  is smooth. We can apply  $\frac{d}{ds}(g \circ \mathfrak{F}^s) = \{g, \chi\} \circ \mathfrak{F}^s$  to a Taylor expansion with reminder

$$\mathbf{G} := \int_0^1 \frac{(1-s)^{n_0}}{n_0!} \mathcal{L}_\chi^{n_0+1}(g) \circ \mathfrak{F}_s ds. \quad (3.11)$$

$\mathcal{L}_\chi^{n_0+1}(g)$  is a linear combination for  $k + \ell = n_0 + 1$  of

$$\mathcal{L}_\chi^\ell(z^\mu \bar{z}^\nu \langle \Phi, f \rangle^\alpha \langle \Psi, \bar{f} \rangle^\beta) \langle \mathcal{L}_\chi^k(f^a \bar{f}^b), \Psi_{ab} \rangle. \quad (3.12)$$

By Lemma 3.1 we have

$$|\mathcal{L}_\chi^\ell(z^\mu \bar{z}^\nu \langle \Phi, f \rangle^\alpha \langle \Psi, \bar{f} \rangle^\beta)| \leq C(|z| + \|f\|_{H^{-K', -S'}})^{2M_0(\ell+1)+2L-a-b} \quad (3.13)$$

and

$$\langle \mathcal{L}_\chi^k(f^a \bar{f}^b), \Psi_{ab} \rangle = \sum_{\sum k_j + \sum k'_j = k} c_{k_1 \dots k_a k'_1 \dots k_b} \langle \prod_{j=1}^a \mathcal{L}_\chi^{k_j}(f) \prod_{j=1}^b \mathcal{L}_\chi^{k'_j}(\bar{f}), \Psi_{ab} \rangle. \quad (3.14)$$

It is easy to conclude that the largest terms in (3.12) are those with  $k = 0$  and  $\ell = n_0 + 1$ . This yields (3.10). When  $g = \langle f^a \bar{f}^b, \Psi_{ab} \rangle$ , the worst terms are the ones of the form

$$\begin{aligned}
|\langle \mathcal{L}_\chi^{n_0+1}(f) f^{a-1} \bar{f}^b, \Psi_{ab} \rangle| &\leq C \sum_{\substack{|\mu|=M_0 \\ |\mu|=|\nu|+1}} |\mathcal{L}_\chi^{n_0}(z^\mu \bar{z}^\nu)| \|f\|_{H^1}^{a+b-1} \leq \\
&\leq C \|f\|_{H^1}^{a+b-1} (|z| + \|f\|_{H^{-K'}, -s'})^{2M_0 n_0 + 2M_0 + 1} \\
&\leq C (|z| + \|f\|_{H^1})^{a+b} (|z| + \|f\|_{H^{-K'}, -s'})^{2M_0(n_0+1)}.
\end{aligned} \tag{3.15}$$

□

### 3.2 Normal forms

We set  $\lambda = (0, \lambda_1, \dots, \lambda_n)$ .

**Definition 3.4.** A function  $Z(t, z, \bar{z}, f, \bar{f})$  is in normal form if it is of the form

$$Z(t, z, \bar{z}, f, \bar{f}) = Z_0(z, \bar{z}) + Z_1(t, z, \bar{z}, f, \bar{f}) \tag{3.16}$$

where we assume properties (N0)–(N2) listed now.

(N0)  $Z_0(z, \bar{z})$  is a finite sum

$$Z_0(z, \bar{z}) = \sum_{\mu\nu} a_{\mu\nu} z^\mu \bar{z}^\nu \text{ with } a_{\mu\nu} \in \mathbb{C} \text{ and with} \tag{3.17}$$

$$\lambda \cdot (\mu - \nu) = 0 \text{ and } |\mu| = |\nu|. \tag{3.18}$$

(N1)  $Z_1(t, z, \bar{z}, f, \bar{f})$  is a finite sum

$$\sum_{m, \mu, \nu} e^{itm} z^\mu \bar{z}^\nu \langle \Phi_{m\mu\nu}, f \rangle + \sum_{m', \mu', \nu'} e^{itm'} z^{\mu'} \bar{z}^{\nu'} \langle \Psi_{m'\mu'\nu'}, \bar{f} \rangle \tag{3.19}$$

with  $\Phi_{m\mu\nu}(x), \Psi_{m'\mu'\nu'}(x) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$  and indexes satisfying

$$\lambda \cdot (\mu - \nu) - m < -\underline{c}, \quad |\mu| = |\nu| - 1, \quad |m| \leq |\mu| \tag{3.20}$$

$$\lambda \cdot (\mu' - \nu') - m' > \underline{c}, \quad |\mu'| = |\nu'| + 1, \quad |m| \leq |\nu'|. \tag{3.21}$$

(N2)  $Z_0(z, \bar{z})$  and  $Z_1(t, z, \bar{z}, f, \bar{f})$  are real valued when  $\bar{z}$  (resp.  $\bar{f}$ ) is the complex conjugate of  $z$  (resp.  $f$ ), that is

$$\bar{a}_{\mu\nu} = a_{\nu\mu}, \quad \Psi_{m\mu\nu}(x) = \bar{\Phi}_{-m\nu\mu}(x). \tag{3.22}$$

We will use the following table of formulas:

$$\begin{aligned}
\{H_F, e^{itm} z^\mu \bar{z}^\nu\} &= i(\lambda \cdot (\mu - \nu) - m) e^{itm} z^\mu \bar{z}^\nu, \\
\{H_F, e^{itm} z^\mu \bar{z}^\nu \langle \Phi, f \rangle\} &= i e^{itm} z^\mu \bar{z}^\nu \langle (\mathcal{H} - \lambda \cdot (\nu - \mu) - m) \Phi, f \rangle, \\
\{H_F, e^{itm'} z^{\mu'} \bar{z}^{\nu'} \langle \Psi, \bar{f} \rangle\} &= -i e^{itm'} z^{\mu'} \bar{z}^{\nu'} \langle (\mathcal{H} - \lambda \cdot (\mu' - \nu') + m') \Psi, \bar{f} \rangle.
\end{aligned} \tag{3.23}$$

We set

$$R_{m\mu\nu} = R_{\mathcal{H}}(\lambda \cdot (\mu - \nu) - m) \tag{3.24}$$

An immediate consequence of the table (3.23) is the following lemma.

**Lemma 3.5.** *Consider finite sums in  $m$ ,  $\mu$  and  $\nu$*

$$K = \sum e^{itm} z^\mu \bar{z}^\nu (k_{m\mu\nu} + \langle \Phi_{m\mu\nu}, f \rangle + \langle \Psi_{m\mu\nu}, \bar{f} \rangle). \tag{3.25}$$

Suppose that  $k_{0\mu\nu} = 0$  if  $(\mu, \nu)$  satisfies (3.18),  $\Phi_{m\mu\nu} = 0$  if  $(m, \mu, \nu)$  satisfies (3.20),  $\Psi_{m'\mu'\nu'} = 0$  if  $(m', \mu', \nu')$  satisfies (3.21). Consider

$$\begin{aligned}
\chi &= i \sum e^{itm} z^\mu \bar{z}^\nu \left[ \langle R_{-m\nu\mu} \Phi_{m\mu\nu}, f \rangle + \right. \\
&\quad \left. \frac{k_{m\mu\nu}}{(\lambda \cdot (\mu - \nu) - m)} - \langle R_{m\mu\nu} \Psi_{m\mu\nu}, \bar{f} \rangle \right].
\end{aligned} \tag{3.26}$$

Then we have

$$\{\chi, H_F\} = K. \tag{3.27}$$

If the coefficients of  $K$  satisfy (3.22), that is

$$\bar{k}_{m\mu\nu} = k_{-m\nu\mu}, \quad \bar{\Phi}_{m\mu\nu} = \Psi_{-m\nu\mu}, \tag{3.28}$$

then also  $\chi$  satisfies analogous equalities.

Finally, we are able to discuss the main result of section 3.

**Theorem 3.6.** *For any integer  $N+1 \geq r \geq 1$  there are a neighborhood  $\mathcal{U}$  of the origin in  $H^1(\mathbb{R}^3, \mathbb{C})$  and a smooth canonical transformation  $\mathcal{T}_r : \mathcal{U} \rightarrow H^1(\mathbb{R}^3, \mathbb{C})$  s.t.*

$$H^{(r)} := H \circ \mathcal{T}_r = H_F + Z^{(r)} + \mathcal{R}^{(r)}. \tag{3.29}$$

where:

- (i)  $Z^{(r)}$  is in normal form, i.e. satisfies (N0)–(N2), with  $2r$ -degree monomials;
- (ii)  $\mathcal{T}_r = \mathcal{T}_{r-1} \circ \mathfrak{F}_r$ , with  $\mathcal{T}_1$  the identity and  $\mathfrak{F}_r$  a transformation as in §3.1 arising from a polynomial  $\chi_r$  as in (3.1);
- (iii) we have  $\mathcal{R}^{(r)} = \sum_{d=0}^7 \mathcal{R}_d^{(r)}$  with the following properties:

(iii.0) we have a finite sum

$$\mathcal{R}_0^{(r)}(t, z) = \sum_{\substack{|\mu|=|\nu|\geq r+1 \\ |m|\leq |\mu|}} a_{m\mu\nu}^{(r)} e^{imt} z^\mu \bar{z}^\nu \quad (3.30)$$

with  $a_{m\mu\nu}^{(r)} \in \mathbb{C}$  s.t.  $\overline{a_{m\mu\nu}^{(r)}} = a_{-m\nu\mu}^{(r)}$ ;

(iii.1) we have a finite sum  $\mathcal{R}_1^{(r)}(t, z, f) =$

$$\sum_{\substack{|\mu|=|\nu|-1\geq r \\ |m|\leq |\mu|}} e^{imt} z^\mu \bar{z}^\nu \langle \Phi_{m\mu\nu}^{(r)}, f \rangle + \sum_{\substack{|\nu|=|\mu|-1\geq r \\ |m|\leq |\nu|}} e^{imt} z^\mu \bar{z}^\nu \langle \Psi_{m\mu\nu}^{(r)}, \bar{f} \rangle \quad (3.31)$$

with  $\Phi_{m\mu\nu}^{(r)} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$  and with  $\Psi_{-m\nu\mu}^{(r)} = \overline{\Phi_{m\mu\nu}^{(r)}}$ ;

(iii.2-5) for  $d = 2, \dots, 5$  we have  $\mathcal{R}_d^{(r)}(t, z, f) =$

$$\sum_{\substack{a+b=d \\ |\mu|+|\alpha|+a= \\ |\nu|+|\beta|+b=L+1\geq 2 \\ |m|\leq L}} e^{imt} z^\mu \bar{z}^\nu \langle \Phi^{(r)}, f \rangle^\alpha \langle \Psi^{(r)}, \bar{f} \rangle^\beta \langle f^a \bar{f}^b, \Psi_{abm\mu\nu\alpha\beta}^{(r)} \rangle,$$

with functions  $\Phi_j^{(r)}$ ,  $\Psi_j^{(r)}$  and  $\Psi_{abm\mu\nu\alpha\beta}^{(r)}$  in  $\mathcal{S}(\mathbb{R}^3, \mathbb{C})$  and with  $\mathcal{R}_d^{(r)}$  real valued;

(iii.6)  $\mathcal{R}_6^{(r)}(t, z, f) = \gamma(t) \int_{\mathbb{R}^3} |f(x)|^4 dx / 4$ ;

(iii.7) we have  $\mathcal{R}_7^{(r)} \in C^\infty(\mathcal{U}^{-K', -S'}, \mathbb{R})$  for all  $(K', S')$  with  $\mathcal{U}^{-K', -S'} = \tilde{\mathcal{U}}^{-K', -S'} \times \mathbb{R}$ , with  $\tilde{\mathcal{U}}^{-K', -S'} \subset \mathbb{C}^{n+1} \times H_c^{-K', -S'}$  an appropriately small neighborhood of the origin, and with

$$|\mathcal{R}_7^{(r)}(t, z, f)| \leq C(|z| + \|f\|_{H^{-K', -S'}})^{2N+4}.$$

*Proof.* Case  $r = 1$  is true with  $Z^{(1)} = 0$ ,  $\mathcal{R}^{(1)} = E_P$  and  $|m| \leq 1$ . We proceed by induction. Set

$$\begin{aligned} \tilde{K}_{r+1} &:= \tilde{\mathcal{R}}_0^{(r)} + \tilde{\mathcal{R}}_1^{(r)}, \\ \tilde{\mathcal{R}}_0^{(r)} &:= \sum_{\substack{|\mu|=|\nu|=r+1 \\ |m|\leq r}} a_{m\mu\nu}^{(r)} e^{imt} z^\mu \bar{z}^\nu \\ \tilde{\mathcal{R}}_1^{(r)} &:= \sum_{\substack{|\mu|=|\nu|-1=r \\ |m|\leq r}} e^{imt} z^\mu \bar{z}^\nu \langle \Phi_{m\mu\nu}^{(r)}, f \rangle + \sum_{\substack{|\nu|=|\mu|-1=r \\ |m|\leq r}} e^{imt} z^\mu \bar{z}^\nu \langle \Psi_{m\mu\nu}^{(r)}, \bar{f} \rangle, \end{aligned} \quad (3.32)$$

i.e.  $\tilde{\mathcal{R}}_0^{(r)}$  (resp.  $\tilde{\mathcal{R}}_1^{(r)}$ ) defined as the sum of the terms in (3.30) with  $|\mu| = r$ , (resp. terms in (3.31) with  $|\mu| + |\nu| = 2r + 1$ ). Split  $\tilde{K}_{r+1} = K_{r+1} + Z_{r+1}$ ,

collecting inside  $Z_{r+1}$  all the terms of  $\tilde{K}_{r+1}$  in the form either (3.17)–(3.18) or (3.19)–(3.21). Apply Lemma 3.5 with  $\chi_{r+1}$  defined from  $K_{r+1}$  in the way (3.26) is defined from (3.25). Then,

$$\{H_F, \chi_{r+1}\} = -K_{r+1}. \quad (3.33)$$

$\chi_{r+1}$  is of form (3.1) with  $m_0 = M_0 = r$ . We can apply Lemmas 3.1–3.3 to  $\chi_{r+1}$ . Let  $\mathfrak{F}_{r+1}$  be as  $\mathfrak{F}$  in (3.3). For  $\mathcal{T}_{r+1} = \mathcal{T}_r \circ \mathfrak{F}_{r+1}$  set

$$H^{(r+1)} := H^{(r)} \circ \mathfrak{F}_{r+1} = H \circ (\mathcal{T}_r \circ \mathfrak{F}_{r+1}) = H \circ \mathcal{T}_{r+1}. \quad (3.34)$$

Split

$$H^{(r)} \circ \phi_{r+1} = H_F + Z^{(r)} + Z_{r+1} \quad (3.35)$$

$$+ (Z^{(r)} \circ \mathfrak{F}_{r+1} - Z^{(r)}) \quad (3.36)$$

$$+ K_{r+1} \circ \mathfrak{F}_{r+1} - K_{r+1} \quad (3.37)$$

$$+ H_F \circ \mathfrak{F}_{r+1} - (H_F + \{H_F, \chi_{r+1}\}) \quad (3.38)$$

$$+ (\mathcal{R}_0^{(r)} - \tilde{\mathcal{R}}_0^{(r)} + \mathcal{R}_1^{(r)} - \tilde{\mathcal{R}}_1^{(r)}) \circ \mathfrak{F}_{r+1} \quad (3.39)$$

$$+ (\mathcal{R}_2^{(r)} + \dots + \mathcal{R}_5^{(r)}) \circ \mathfrak{F}_{r+1} \quad (3.40)$$

$$+ \mathcal{R}_6^{(r)} \circ \mathfrak{F}_{r+1} \quad (3.41)$$

$$+ \mathcal{R}_7^{(r)} \circ \mathfrak{F}_{r+1}. \quad (3.42)$$

Define  $Z^{(r+1)} := Z^{(r)} + Z_{r+1}$ . Then it is a degree  $2r+2$  polynomial in  $(z, \bar{z}, f, \bar{f})$  of the form (3.16) satisfying (3.17)–(3.22).  $\mathcal{R}_7^{(r)} \circ \mathfrak{F}_{r+1}$  can be absorbed in  $\mathcal{R}_7^{(r+1)}$ . For (3.36)–(3.41) we apply Lemma 3.3 with  $n_0 + 1 = N + 2$ . The reminder terms corresponding to  $\mathbf{G}$  in (3.10) can be absorbed in  $\mathcal{R}_7^{(r+1)}$ . To all the terms corresponding to  $\mathcal{V}$  in (3.10) we can apply (b) Lemma 3.3. In particular, they are of the form (3.9) with  $L' = L + lr$  and with  $|m'| \leq L + lr$ . Since  $L \geq 1$ , this means they are of degree at least  $2r+4$  and they can be absorbed in  $\mathcal{R}^{(r+1)}$ .  $\square$

### 3.3 Application of Theorem 3.6

We apply Theorem 3.6 for  $r = N + 1$ . Hence we have  $H^{(r)} := H \circ \mathcal{T}_r = H_F + Z_0^{(r)} + Z_1^{(r)} + \mathcal{R}^{(r)}$ . To simplify notation, but not only for this reason, we will drop the super indexes but not in an obvious way. The first step is obvious. We set

$$Z_0 = Z_0^{(r)}. \quad (3.43)$$

To proceed, we go back to the normal forms in (N1) and to conditions (3.19)–(3.22). Set

$$\begin{aligned} \mathbf{M} &= \{(m, \mu, \nu) : \lambda \cdot (\mu - \nu) - m < -\underline{c}, |\mu| = |\nu| - 1, |m| \leq |\mu| \leq N\} \\ \mathbf{M}' &= \{(m', \mu', \nu') : (-m', \nu', \mu') \in \mathbf{M}\}. \end{aligned} \quad (3.44)$$

Notice that the two inequalities  $|\mu| = |\nu| - 1$  and  $|m| \leq |\mu| \leq N$ , by hypothesis (H7) imply  $\lambda \cdot (\nu - \mu) + m \neq \underline{c}$ .

We have

$$Z_1^{(r)} = \sum_{(m,\mu,\nu) \in \mathbf{M}} e^{imt} z^\mu \bar{z}^\nu \langle \Phi_{m\mu\nu}^{(r)}, f \rangle + \sum_{(m',\mu',\nu') \in \mathbf{M}'} e^{im't} z^{\mu'} \bar{z}^{\nu'} \langle \Psi_{m'\mu'\nu'}^{(r)}, \bar{f} \rangle.$$

By (iii.0)–(iii.1) Theorem 3.6

$$\overline{\Phi^{(r)}}_{m\mu\nu} = \Psi_{-m\nu\mu}^{(r)}. \quad (3.45)$$

**Definition 3.7.** Denote by  $M$  the subset of the indexes  $(m, \mu, \nu) \in \mathbf{M}$  with the following property: if  $(m, \alpha, \beta) \in \mathbf{M}$  and if  $\alpha_j \leq \mu_j$  and  $\beta_j \leq \nu_j$  for  $j = 0, \dots, n$ , then  $(\alpha, \beta) = (\mu, \nu)$ .

**Definition 3.8.** Denote by  $M'$  the subset of the indexes  $(m', \mu', \nu') \in \mathbf{M}'$  with the following property: if  $(m', \alpha', \beta') \in \mathbf{M}'$  and if  $\alpha'_j \leq \mu'_j$  and  $\beta'_j \leq \nu'_j$  for  $j = 0, \dots, n$ , then  $(\alpha', \beta') = (\mu', \nu')$ .

The following is a straightforward consequence of Theorem 3.6:

**Lemma 3.9.** *The following is a 1-1 and onto map  $M \rightarrow M'$ :*

$$(m, \mu, \nu) \rightarrow (m', \mu', \nu') \text{ where } m' = -m, \mu' = \nu, \nu' = \mu. \quad (3.46)$$

We set, and this is non obvious,

$$Z_1 = \sum_{(m,\mu,\nu) \in M} e^{imt} z^\mu \bar{z}^\nu \langle \Phi_{m\mu\nu}^{(r)}, f \rangle + \sum_{(m',\mu',\nu') \in M'} e^{im't} z^{\mu'} \bar{z}^{\nu'} \langle \Psi_{m'\mu'\nu'}^{(r)}, \bar{f} \rangle \quad (3.47)$$

and

$$\mathcal{R} = \mathcal{R}^{(r)} + Z_1^{(r)} - Z_1. \quad (3.48)$$

Finally, to simplify notation, we set

$$\Phi_{m\mu\nu} = \Phi_{m\mu\nu}^{(r)}, \quad \Psi_{m\mu\nu} = \Psi_{m\mu\nu}^{(r)}. \quad (3.49)$$

## 4 Dispersion

We apply Theorem 3.6 for  $r = N + 1$  and we use the notation in Subsection 3.3. We will show:

**Theorem 4.1.** *There is a fixed  $C > 0$  such that for  $\varepsilon_0 > 0$  sufficiently small and for  $\varepsilon \in (0, \varepsilon_0)$  we have*

$$\|f\|_{L_t^r([0,\infty), W_x^{1,p})} \leq C\varepsilon \text{ for all admissible pairs } (r, p) \quad (4.1)$$

$$\|z^{\mu+\nu}\|_{L_t^2([0,\infty))} \leq C\varepsilon \text{ for all } (m, \mu, \nu) \in M \quad (4.2)$$

$$\|z_j\|_{W_t^{1,\infty}([0,\infty))} \leq C\varepsilon \text{ for all } j \in \{0, \dots, n\}. \quad (4.3)$$

By Theorem 3.6, Theorem 4.1 implies Theorem 1.1. Notice that (1.1) is time reversible, so in particular (4.1)–(4.3) are true over the whole real line. The proof, though, exploits that  $t \geq 0$ , specifically when for  $\lambda \in \sigma_c(\mathcal{H})$  we choose  $R_{\mathcal{H}}^+(\lambda) = R_{\mathcal{H}}(\lambda + i0)$  rather than  $R_{\mathcal{H}}^-(\lambda) = R_{\mathcal{H}}(\lambda - i0)$  in formula (4.10). See the discussion on p.18 [SW].

The proof of Theorem 4.1 is a standard continuation argument. We assume

$$\|f\|_{L_t^r([0,T], W_x^{1,p})} \leq C_1 \epsilon \text{ for all admissible pairs } (r, p) \quad (4.4)$$

$$\|z^{\mu+\nu}\|_{L_t^2([0,T])} \leq C_2 \epsilon \text{ for all } (m, \mu, \nu) \in M \quad (4.5)$$

$$\|z_j\|_{W_t^{1,\infty}([0,T])} \leq C_3 \epsilon \text{ for all } j \in \{1, \dots, m\} \quad (4.6)$$

for fixed sufficiently large constants  $C_1, C_2$  and  $C_3$ . Then we prove that for  $\epsilon$  sufficiently small, (4.4)–(4.6) imply the same estimate but with  $C_1, C_2, C_3$  replaced by  $C_1/2, C_2/2, C_3/2$ . Then (4.4)–(4.6) hold with  $[0, T]$  replaced by  $[0, \infty)$ .

The proof consists in three main steps.

- (i) Estimate  $f$  in terms of  $z$ .
- (ii) Substitute the variable  $f$  with a new "smaller" variable  $g$  and find smoothing estimates for  $g$ .
- (iii) Reduce the system for  $z$  to a closed system involving only the  $z$  variables, by insulating the part of  $f$  which interacts with  $z$ , and by decoupling the rest (this remainder is  $g$ ). Then clarify the nonlinear Fermi golden rule.

These three steps are the same of the material in [BC] from Section 7 on, and [C1] from Section 10 on. We start by sketching steps (i) and (ii). Step (i) is encapsulated by the following proposition:

**Proposition 4.2.** *Assume (4.4)–(4.6). Then there exist constants  $K_1$  and  $C = C(C_1, C_2, C_3)$ , with  $K_1$  independent of  $C_1$ , such that, if  $C\epsilon$  is sufficiently small, then we have*

$$\|f\|_{L_t^r([0,T], W_x^{1,p})} \leq K_1 \epsilon \text{ for all admissible pairs } (r, p). \quad (4.7)$$

Consider  $Z_1$  of the form (3.19)–(3.22). Then we have (with finite sums)

$$i\dot{f} - \mathcal{H}f = \sum_{(m', \mu', \nu') \in M'} e^{itm'} z^{\mu'} \bar{z}^{\nu'} \Psi_{m'\mu'\nu'}(x) + \nabla_{\bar{f}} \mathcal{R}. \quad (4.8)$$

The proof of Proposition 4.2 is standard and we skip it, see [CM]. The dominating term in the rhs of (4.8) is the first line in the rhs. Notice also, that Theorem 4.1 implies by standard arguments, see [CM],

$$\lim_{t \rightarrow +\infty} \|f(t) - e^{it\Delta\sigma_3} f_+\|_{H^1} = 0 \quad (4.9)$$

for a  $f_+ \in H^1$  with  $\|f_+\|_{H^1} \leq C\epsilon$  and for a real valued function  $\theta \in C^1(\mathbb{R}, \mathbb{R})$ .



Step (ii) in the proof of Theorem 4.1 consists in introducing the variable

$$g = f + \sum_{(m', \mu', \nu') \in M'} e^{itm'} z^{\mu'} \bar{z}^{\nu'} R_{\mathcal{H}}^+(\lambda \cdot (\mu' - \nu') - m') \Psi_{m', \mu', \nu'}(x). \quad (4.10)$$

Notice that by Lemma 7 ch. XIII.8 [RS], we have  $g \in L_x^{2, -S}(\mathbb{R}^3)$  for  $S > 1/2$ . Substituting the new variable  $g$  in (4.8), the first line on the rhs of (4.8) cancels out. By an easier version of Lemma 4.3 [CM] we have:

**Lemma 4.3.** *For  $\epsilon$  sufficiently small and for  $S > 0$  sufficiently large, there exists  $C_0 = C_0(\mathcal{H}, S)$  a fixed constant such that*

$$\|g\|_{L_t^2 L_x^{2, -S}} \leq C_0 \epsilon + O(\epsilon^2). \quad (4.11)$$

We skip the proof, which is standard, see [CM]. As in [BP, SW] and subsequent literature, the part of  $f$  which couples nontrivially with  $z$  comes from the polynomial in  $z$  in the summation in rhs(4.10). In some sense,  $g$  and  $z$  decouple.

#### 4.1 The Fermi golden rule

We proceed like [BC, C1], with in (4.30) a different Lyapunov functional than in [BC, C1]. Set  $R_{\mu\nu m}^\pm = R_{\mathcal{H}}^\pm(\lambda \cdot (\mu - \nu) - m)$ . System (2.16) with the new hamiltonian  $H^{(r)}$  yields (notice that in the first line in (4.12) summation is over  $M$ , while in the second line line in (4.12) is over  $M'$ )

$$\begin{aligned} i\dot{z}_j - \lambda_j z_j &= \partial_{\bar{z}_j} Z_0(z) + \sum_{(m, \mu, \nu) \in M} \nu_j e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \langle f, \Phi_{m\mu\nu} \rangle \\ &+ \sum_{(m, \mu, \nu) \in M'} \nu_j e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \langle \bar{f}, \Psi_{m\mu\nu} \rangle + \partial_{\bar{z}_j} \mathcal{R}. \end{aligned} \quad (4.12)$$

We substitute (4.10) in (4.12). Then we rewrite (4.12)

$$i\dot{z}_j - \lambda_j z_j = \partial_{\bar{z}_j} Z_0(z) + \mathcal{E}_j - \quad (4.13)$$

$$- \sum_{\substack{(m, \mu, \nu) \in M \\ (m', \mu', \nu') \in M'}} \nu_j e^{i(m+m')t} \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{\bar{z}_j} \langle R_{\mu'\nu'm'}^+ \Psi_{m'\mu'\nu'}, \Phi_{m\mu\nu} \rangle \quad (4.14)$$

$$- \sum_{\substack{(m, \mu, \nu) \in M' \\ (m', \mu', \nu') \in M'}} \nu_j e^{i(m-m')t} \frac{z^{\mu+\nu'} \bar{z}^{\nu+\mu'}}{\bar{z}_j} \langle R_{\mu'\nu'm'}^- \overline{\Psi_{m'\mu'\nu'}}, \Psi_{m\mu\nu} \rangle \quad (4.15)$$

with  $\mathcal{E}_j := \text{rhs}(4.12) - (4.14) - (4.15)$ . Most terms in (4.14)–(4.15) can be eliminated through new variables. Summing up only on the subsets  $\mathcal{M}_1 \subseteq M \times M'$  and  $\mathcal{M}_2 \subseteq M' \times M'$ , defined by the fact that the denominators in (4.16) are non zero, we set

$$\begin{aligned} z_j &= \zeta_j + \sum_{\mathcal{M}_1} \frac{\nu_j e^{i(m+m')t} z^{\mu+\mu'} \bar{z}^{\nu+\nu'} \langle R_{\mu'\nu'm'}^+ \Psi_{m'\mu'\nu'}, \Phi_{m\mu\nu} \rangle}{(m+m' - \lambda \cdot (\mu + \mu' - \nu - \nu')) \bar{z}_j} \\ &+ \sum_{\mathcal{M}_2} \frac{\nu_j e^{i(m-m')t} z^{\mu+\nu'} \bar{z}^{\nu+\mu'} \langle R_{\mu'\nu'm'}^- \overline{\Psi_{m'\mu'\nu'}}, \Psi_{m\mu\nu} \rangle}{(m-m' - \lambda \cdot (\mu + \nu' - \nu - \mu')) \bar{z}_j}. \end{aligned} \quad (4.16)$$

**Lemma 4.4.** *In (4.16) we have contributions from all terms in (4.14) (resp. (4.15)) with  $m \neq -m'$  (resp.  $m \neq m'$ ).*

*Proof.* We consider only contributions from (4.14). Contributions from (4.15) can be treated similarly. It is enough to show that for  $m \neq -m'$ ,  $(m, \mu, \nu) \in \mathbf{M}$  and  $(m', \mu', \nu') \in \mathbf{M}'$ , we have

$$m + m' - \lambda \cdot (\mu + \mu' - \nu - \nu') \neq 0. \quad (4.17)$$

By the definition of  $\mathbf{M}$  and  $\mathbf{M}'$  in (3.44), we have  $|\mu + \mu' - \nu - \nu'| \leq 4N + 2$  and  $|m + m'| \leq 2N$ . Then lhs(4.17) = 0 by hypothesis (H8) would imply  $m + m' = 0$ . This proves (4.17).  $\square$

By (4.5)–(4.6) and by the fact that in (4.16) we have  $|\nu| > 1$ , we obtain:

$$\begin{aligned} \|\zeta - z\|_{L_t^2} &\leq \tilde{C}C_3\epsilon \sum_{(m', \mu', \nu') \in M'} \|z^{\mu'+\nu'}\|_{L_t^2} \leq CC_2C_3\epsilon^2, \\ \|\zeta - z\|_{L_t^\infty} &\leq CC_3^3\epsilon^3, \end{aligned} \quad (4.18)$$

with  $C$  a fixed constant. In the new variables, equation (4.13)–(4.15) is of the form ( $\mathcal{D}_j$  is discussed later, in the course of the proof of Lemma 4.8)

$$\begin{aligned} i\dot{\zeta}_j - \lambda_j \zeta_j - \partial_{\bar{\zeta}_j} Z_0(\zeta) - \mathcal{D}_j = \\ - \sum_{\substack{(m, \mu, \nu) \in M \\ (-m, \mu', \nu') \in M' \\ \lambda \cdot (\mu + \mu') = \lambda \cdot (\nu + \nu')}} \nu_j \frac{\zeta^{\mu+\mu'} \bar{\zeta}^{\nu+\nu'}}{\bar{\zeta}_j} \langle R_{\mu'\nu'(-m)}^+ \overline{\Phi_{m\nu'\mu'}}, \Phi_{m\mu\nu} \rangle - \\ - \sum_{\substack{(m, \mu, \nu) \in M' \\ (m, \mu', \nu') \in M' \\ \lambda \cdot (\mu - \nu) = \lambda \cdot (\mu' - \nu')}} \nu_j \frac{\zeta^{\mu+\nu'} \bar{\zeta}^{\nu+\mu'}}{\bar{\zeta}_j} \langle R_{\mu'\nu'm}^- \overline{\Psi_{m\mu'\nu'}}, \Psi_{m\mu\nu} \rangle. \end{aligned} \quad (4.19)$$

Set  $\alpha = \nu'$ ,  $\beta = \mu'$  in the first term in rhs(4.19); replace in the second term in rhs(4.19),  $\alpha = \nu'$ ,  $\beta = \mu'$ ,  $(\mu, \nu)$  with  $(\nu, \mu)$  and  $m$  with  $-m$ . Then we get:

$$\begin{aligned} i\dot{\zeta}_j - \lambda_j \zeta_j - \partial_{\bar{\zeta}_j} Z_0(\zeta) - \mathcal{D}_j = \\ - \sum_{\substack{(m, \mu, \nu) \in M \\ (m, \alpha, \beta) \in M \\ \lambda \cdot (\mu - \nu) = \lambda \cdot (\alpha - \beta)}} \nu_j \frac{\zeta^{\mu+\beta} \bar{\zeta}^{\nu+\alpha}}{\bar{\zeta}_j} \langle R_{\mathcal{H}}^+(\lambda \cdot (\beta - \alpha) + m) \overline{\Phi_{m\alpha\beta}}, \Phi_{m\mu\nu} \rangle - \\ - \sum_{\substack{(m, \mu, \nu) \in M \\ (m, \alpha, \beta) \in M \\ \lambda \cdot (\mu - \nu) = \lambda \cdot (\alpha - \beta)}} \mu_j \frac{\zeta^{\nu+\alpha} \bar{\zeta}^{\mu+\beta}}{\bar{\zeta}_j} \langle R_{\mathcal{H}}^-(\lambda \cdot (\beta - \alpha) + m) \overline{\Phi_{m\alpha\beta}}, \Phi_{m\mu\nu} \rangle. \end{aligned} \quad (4.20)$$

Let now  $X = \{\lambda \cdot (\beta - \alpha) + m : (m, \alpha, \beta) \in M\}$ .  $M$  is a finite set, so also  $X$  is a finite set. For each  $w \in X$  let  $M_w = \{(m, \alpha, \beta) \in M : \lambda \cdot (\beta - \alpha) + m = w\}$ .

*Remark 4.5.* Notice that if  $(m, \mu, \nu) \in M_w$  and  $(m', \mu', \nu') \in M_w$ , then  $m = m'$  and  $\lambda \cdot (\mu - \nu) = \lambda \cdot (\mu' - \nu')$  by (H8).

Recall Plemelj formula  $\frac{1}{x \pm i0} = P.V. \frac{1}{x} \mp i\pi\delta(x)$ . Then, proceeding as in [SW, BC] in this spot, we write

$$\begin{aligned} i\dot{\zeta}_j - \lambda_j \zeta_j - \partial_{\bar{\zeta}_j} Z_0(\zeta) - \mathcal{D}_j = \\ - \sum_{w \in X} \sum_{\substack{(m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} \frac{\nu_j + \alpha_j}{\bar{\zeta}_j} \langle P.V. \frac{1}{\mathcal{H} - w} \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle \\ + i\pi \sum_{w \in X} \sum_{\substack{(m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} \frac{\alpha_j - \nu_j}{\bar{\zeta}_j} \langle \delta(\mathcal{H} - w) \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle. \end{aligned} \quad (4.21)$$

Here we take a minor departure from [BC]. We prove:

**Lemma 4.6.** *Let*

$$\Phi_w := \sum_{(m, \mu, \nu) \in M_w} \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu}. \quad (4.22)$$

*Then, we have*

$$\sum_{j=0}^n \frac{1}{2} \partial_t |\zeta_j|^2 + \pi \sum_{w \in X} \langle \delta(\mathcal{H} - w) \overline{\Phi_w}, \Phi_w \rangle = \sum_{j=0}^n \text{Im}(\mathcal{D}_j \bar{\zeta}_j). \quad (4.23)$$

*Proof.* We multiply (4.21) by  $\bar{\zeta}_j$  and sum on  $j = 0, \dots, n$  and take the imaginary part of the sum. Then, (4.23) is an immediate consequence of two cancelations, (4.24) and (4.26) below, and one identity, (4.27) below. To finish the proof of Lemma 4.6 we need to state and prove (4.24), (4.26) and (4.27). By (3.17)–(3.18) and by  $Z_0 = \overline{Z_0}$  we get the the first cancelation:

$$\begin{aligned} 2i \sum_{j=0}^n \text{Im}(\bar{\zeta}_j \partial_{\bar{\zeta}_j} Z_0(\zeta)) &= \sum_{j=0}^n (\bar{\zeta}_j \partial_{\bar{\zeta}_j} Z_0(\zeta) - \zeta_j \partial_{\zeta_j} Z_0(\zeta)) \\ &= \sum_{j=0}^n \sum_{|\mu|=|\nu|} (\mu_j - \nu_j) a_{\mu\nu} \zeta^\mu \bar{\zeta}^\nu = \sum_{|\mu|=|\nu|} (|\mu| - |\nu|) a_{\mu\nu} \zeta^\mu \bar{\zeta}^\nu = 0. \end{aligned} \quad (4.24)$$

We turn to the second cancelation. For  $(m, \mu, \nu)$  and  $(m, \alpha, \beta)$  elements of  $M_w \subset \mathbf{M}$ , by the definition of  $\mathbf{M}$  in (3.44) we have

$$\sum_{j=0}^n (\mu_j - \nu_j) =: |\mu| - |\nu| = -1 = |\alpha| - |\beta| = \sum_{j=0}^n (\alpha_j - \beta_j). \quad (4.25)$$

By (4.25) we get  $\sum_{j=0}^n (\nu_j + \alpha_j) = |\nu| + |\alpha| = |\beta| + |\mu|$ . Hence

$$\begin{aligned}
& \operatorname{Im} \sum_{\substack{w \in X \\ (m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} (|\nu| + |\alpha|) \langle P.V. \frac{1}{\mathcal{H} - w} \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle = \\
& \operatorname{Im} \sum_{\substack{w \in X \\ (m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} \frac{|\alpha| + |\beta| + |\mu| + |\nu|}{2} \langle P.V. \frac{1}{\mathcal{H} - w} \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle \quad (4.26) \\
& = 0.
\end{aligned}$$

Having stated and proved the second cancelation (4.26), we turn to the last identity, (4.25) below. Rearranging, by  $\operatorname{Re} \langle \delta(\mathcal{H} - w) \bar{v}_1, v_2 \rangle = \operatorname{Re} \langle \delta(\mathcal{H} - w) \bar{v}_2, v_1 \rangle$  and by (4.25), we get

$$\begin{aligned}
& \sum_{\substack{w \in X \\ (m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} (|\alpha| - |\nu|) \operatorname{Re} \langle \delta(\mathcal{H} - w) \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle = \\
& \frac{1}{2} \sum_{\substack{w \in X \\ (m, \mu, \nu) \in M_w \\ (m, \alpha, \beta) \in M_w}} (|\alpha| - |\nu| + |\mu| - |\beta|) \operatorname{Re} \langle \delta(\mathcal{H} - w) \overline{\zeta^\alpha \bar{\zeta}^\beta \Phi_{m\alpha\beta}}, \zeta^\mu \bar{\zeta}^\nu \Phi_{m\mu\nu} \rangle \quad (4.27) \\
& = - \sum_{w \in X} \langle \delta(\mathcal{H} - w) \Phi_w, \Phi_w \rangle.
\end{aligned}$$

This yields Lemma 4.6.  $\square$

*Remark 4.7.* Formula (4.23) with Lemma 4.8 below is the crucial structural result in the paper. This is a form of the so called *nonlinear Fermi golden rule*.

**Lemma 4.8.** *Assume inequalities (4.5)–(4.6). Then for a fixed constant  $c_0$  we have*

$$\sum_{j=0}^n \|\mathcal{D}_j \bar{\zeta}_j\|_{L^1[0, T]} \leq (1 + C_2) c_0 \epsilon^2. \quad (4.28)$$

We postpone the proof, assume the conclusion and complete the proof of Theorem 4.1. We introduce now the following key hypothesis.

(H9') We assume that for some fixed constants for any vector  $\zeta \in \mathbb{C}^n$  we have:

$$\sum_{w \in X} \langle \delta(\mathcal{H} - w) \overline{\Phi_w}, \Phi_w \rangle \approx \sum_{(m, \mu, \nu) \in M} |\zeta^{\mu+\nu}|^2. \quad (4.29)$$

Now we complete the proof of Theorem 4.1. Notice that  $\text{lhs}(4.29) \geq 0$ . By (4.23)–(4.29)

$$\sum_j |z_j(t)|^2 + \sum_{(m,\mu,\nu) \in M} \|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_2 \epsilon^2. \quad (4.30)$$

By (4.18) this implies  $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_2 \epsilon^2$  for all the above multi indexes. So, from  $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim C_2^2 \epsilon^2$  we conclude  $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim C_2 \epsilon^2$ . This means that we can take  $C_2 \approx 1$ . This yields Theorem 4.1.

**Proof of Lemma 4.8.** We have schematically

$$\begin{aligned} \mathcal{D}_j &= \mathcal{E}_j + \bar{\partial}_j Z_0(t, z) - \bar{\partial}_j Z_0(t, \zeta) \\ &+ \sum_k \partial_{z_k} \partial_{\bar{z}_j} \left( \sum_{\mathcal{M}_1} z^{\mu+\mu'} \bar{z}^{\nu+\nu'} + \sum_{\mathcal{M}_2} z^{\mu+\nu'} \bar{z}^{\nu+\mu'} \right) \text{rhs}(4.12)_k \\ &- \sum_k \partial_{\bar{z}_k} \partial_{z_j} \left( \sum_{\mathcal{M}_1} z^{\mu+\mu'} \bar{z}^{\nu+\nu'} + \sum_{\mathcal{M}_2} z^{\mu+\nu'} \bar{z}^{\nu+\mu'} \right) \overline{\text{rhs}(4.12)_k} \end{aligned} \quad (4.31)$$

where the exponents in the second line are the sums of the exponents in (4.16), where  $\text{rhs}(4.12)_k$  is just (4.12) when  $j = k$  and where

$$\begin{aligned} \mathcal{E}_j &= \partial_{\bar{z}_j} \mathcal{R} + \\ &\sum_{(m,\mu,\nu) \in M} \nu_j e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \langle g, \Phi_{m\mu\nu} \rangle + \sum_{(m,\mu,\nu) \in M'} \nu_j e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \langle \bar{g}, \Psi_{m\mu\nu} \rangle \\ &- \sum_{\substack{(m,\mu,\nu) \notin M \\ \lambda \cdot (\mu - \nu) - m < \underline{c} \\ |\mu| = |\nu| - 1 \\ (m', \mu', \nu') \in M'}} \nu_j e^{i(m+m')t} \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{\bar{z}_j} \langle R_{\mu'\nu'm'}^+ \Psi_{m'\mu'\nu'}, \Phi_{m\mu\nu} \rangle \\ &- \sum_{\substack{(m,\mu,\nu) \notin M' \\ \lambda \cdot (\mu - \nu) - m > \underline{c} \\ |\mu| = |\nu| + 1 \\ (m', \mu', \nu') \in M'}} \nu_j e^{i(m-m')t} \frac{z^{\mu+\nu'} \bar{z}^{\nu+\mu'}}{\bar{z}_j} \langle R_{\mu'\nu'm'}^- \overline{\Psi_{m'\mu'\nu'}}, \Psi_{m\mu\nu} \rangle. \end{aligned} \quad (4.32)$$

We now estimate one by one the terms in (4.31). Lemma 4.8 is an immediate consequence of Lemmas 4.9–4.11 below.

**Lemma 4.9.** *There are fixed  $C_0$  and  $\epsilon_0 > 0$  such that for  $\epsilon \in (0, \epsilon_0)$  we have*

$$\begin{aligned} \|\mathcal{E}_j \bar{\zeta}_j\|_{L_t^1[0,T]} &\leq (1 + C_2) C_0 \epsilon^2, \\ \|\mathcal{E}_j\|_{L_t^2[0,T]} &\leq C_0 \epsilon. \end{aligned} \quad (4.33)$$

*Proof.* We have for  $C = C(C_1, C_2, C_3)$

$$\|\bar{\zeta}_j \partial_{\bar{z}_j} \mathcal{R}\|_{L_t^1} \leq \|\partial_{\bar{z}_j} \mathcal{R}\|_{L_t^1} \|\bar{\zeta}_j\|_{L_t^\infty} \leq 2C_3 \epsilon \sum_{d=0}^7 \|\partial_{\bar{z}_j} \mathcal{R}_d\|_{L_t^1} \leq C\epsilon^3 \quad (4.34)$$

by (4.4)–(4.6), (4.18), by Theorem 3.6 for  $r = N + 1$ .

We prove now that, for  $\Phi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ , we have

$$\begin{aligned} & \sum_{(m, \mu, \nu) \in M} \nu_j \|e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \bar{\zeta}_j \langle \Phi, g \rangle\|_{L_t^1} + \\ & \sum_{(m, \mu, \nu) \in M'} \nu_j \|e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \bar{\zeta}_j \langle \Phi, \bar{g} \rangle\|_{L_t^1} \leq (1 + C_2) C_0 \epsilon^2 \end{aligned} \quad (4.35)$$

for a fixed  $C_0$ . These are the terms responsible for  $C_2 C_0 \epsilon^2$  in (4.28). All the other terms can be incorporated in  $C_0 \epsilon^2$ . We focus on

$$e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \bar{\zeta}_j \langle \Phi, g \rangle = e^{imt} z^\mu \bar{z}^\nu \langle \Phi, g \rangle + e^{imt} \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} (\bar{\zeta}_j - \bar{z}_j) \langle \Phi, g \rangle. \quad (4.36)$$

We have for  $(m, \mu, \nu)$  either in  $M$  or in  $M'$  and by Lemma 4.3

$$\|e^{imt} z^\mu \bar{z}^\nu \langle \Phi, g \rangle\|_{L_t^1} \lesssim \|z^\mu \bar{z}^\nu\|_{L_t^2} \|g\|_{L_t^2 L_x^{2, -s}} \lesssim C_2 \epsilon^2. \quad (4.37)$$

Turning to the second term in rhs(4.36)

$$\|z^\mu \frac{\bar{z}^\nu}{\bar{z}_j} (\bar{\zeta}_j - \bar{z}_j) \langle \Phi, g \rangle\|_{L_t^1} \leq \|z^\mu \frac{\bar{z}^\nu}{\bar{z}_j}\|_{L_t^\infty} \|\bar{\zeta}_j - \bar{z}_j\|_{L_t^2} \|g\|_{L_t^2 L_x^{2, -s}} \leq C(C_2, C_3) \epsilon^3$$

by (4.4)–(4.6), by Lemma 4.3 and by (4.18).

We consider now contributions to  $\mathcal{E}_j \bar{\zeta}_j$  coming from the third line of (4.32). Terms from the fourth line of (4.32) can be treated similarly. We can write them

$$\sum_{\substack{(m, \mu, \nu) \notin M \\ \lambda \cdot (\mu - \nu) - m < c \\ |\mu| = |\nu| - 1 \\ (m', \mu', \nu') \in M'}} \nu_j (\|z^{\mu + \mu'} \bar{z}^{\nu + \nu'}\|_{L_t^1} + \|\frac{z^{\mu + \mu'} \bar{z}^{\nu + \nu'}}{\bar{z}_j} (\bar{z}_j - \bar{\zeta}_j)\|_{L_t^1}) \quad (4.38)$$

where we omitted the factors  $e^{i(m+m')t}$ . It is easy to understand that the largest terms in (4.38) are the ones with  $(m, \mu, \nu) \in \mathbf{M} \setminus M$ . For each  $(m, \mu, \nu) \in \mathbf{M} \setminus M$  there exists a  $(m, \alpha, \beta) \in M$  with  $\alpha_j \leq \mu_j$  and  $\beta_j \leq \nu_j$  for all  $j = 0, \dots, n$  and with at least one of these not an equality. Hence

$$\|z^{\mu + \mu'} \bar{z}^{\nu + \nu'}\|_{L_t^1} \leq \|z\|_{L_t^\infty} \|z^{\alpha + \beta}\|_{L_t^2} \|z^{\mu' + \nu'}\|_{L_t^2} \leq C_3 C_2^2 \epsilon^3. \quad (4.39)$$

The second terms in (4.38) can be bounded by

$$\left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{\bar{z}_j} (\bar{z}_j - \bar{\zeta}_j) \right\|_{L_t^1} \leq \left\| \frac{z^{\mu} \bar{z}^{\nu}}{\bar{z}_j} \right\|_{L_t^\infty} \|z^{\alpha+\beta}\|_{L_t^2} \|z - \zeta\|_{L_t^2} \leq C(C_2, C_3) \epsilon^3. \quad (4.40)$$

We have  $\|\mathcal{E}_k\|_{L_t^2} \leq C(C_2, C_3) \epsilon^2$ , as can be easily seen by (4.32) and the estimates used in (4.34).  $\square$

**Lemma 4.10.** *There is a fixed  $\epsilon_0 > 0$  such that assuming (4.4)–(4.6) and for  $\epsilon \in (0, \epsilon_0)$  we have*

$$\|(\bar{\partial}_j Z_0(t, z) - \bar{\partial}_j Z_0(t, \zeta)) \bar{\zeta}_j\|_{L_t^1[0, T]} \leq C(C_2, C_3) \epsilon^3. \quad (4.41)$$

*Proof.* We consider quantities  $(\frac{z^\mu \bar{z}^\nu}{\bar{\zeta}_j} - \frac{z^\mu \bar{z}^\nu}{\bar{z}_j}) \bar{\zeta}_j$  with  $(\mu, \nu)$  s.t.  $|\mu| = |\nu|$  and  $\lambda \cdot (\mu - \nu) = 0$ , see (3.18). By Taylor expansion these are

$$\sum \partial_k \left( \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \right) (\zeta_k - z_k) \bar{\zeta}_j + \sum \bar{\partial}_k \left( \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \right) (\bar{\zeta}_k - \bar{z}_k) \bar{\zeta}_j + \bar{\zeta}_j O(|z - \zeta|^2). \quad (4.42)$$

It is straightforward that by (4.6) and (4.18)

$$\|\bar{\zeta}_j O(|z - \zeta|^2)\|_{L_t^1} \leq C(C_2, C_3) \epsilon^3. \quad (4.43)$$

Turning to the first terms in (4.42) we split

$$\mu_k \frac{z^\mu \bar{z}^\nu}{z_k \bar{z}_j} \bar{\zeta}_j (\zeta_k - z_k) = \mu_k \frac{z^\mu \bar{z}^\nu}{z_k} (\zeta_k - z_k) + \mu_k \frac{z^\mu \bar{z}^\nu}{z_k \bar{z}_j} (\bar{\zeta}_j - \bar{z}_j) (\zeta_k - z_k). \quad (4.44)$$

The second term in rhs(4.44) can be treated like (4.43). To bound the first term in rhs(4.44) we substitute (4.16). We have

$$\mu_k \left\| \frac{z^\mu \bar{z}^\nu}{z_k} (\zeta_k - z_k) \right\|_{L_t^1} \leq \sum_{\mathcal{M}_1 \cup \mathcal{M}_2} \mu_k \beta_k \left\| \frac{z^\mu \bar{z}^\nu}{z_k} e^{i(m \pm m')t} \frac{z^{\alpha+\alpha'} \bar{z}^{\beta+\beta'}}{\bar{z}_k} \right\|_{L_t^1}.$$

We claim that (3.18),  $|\mu| + |\nu| \leq 2N + 2$ , by (i) Theorem 3.6, and hypothesis (H8) imply that there is at least one index  $\nu_\ell \neq 0$  such that  $\lambda_\ell = \lambda_k$ . Indeed, if  $k \geq 1$ ,

$$\lambda_k \sum_{\ell: \lambda_\ell = \lambda_k} (\mu_\ell - \nu_\ell) = 0 = \sum_{\ell: \lambda_\ell = \lambda_k} (\mu_\ell - \nu_\ell). \quad (4.45)$$

This yields the existence of the  $\nu_\ell \neq 0$  for  $k \neq 0$ . (4.45) implies

$$\sum_{\ell \neq 0} (\mu_\ell - \nu_\ell) = 0. \quad (4.46)$$

Finally, (4.46) and  $|\mu| = |\nu|$  imply also  $\mu_0 = \nu_0$ , and our claim for  $k = 0$ . Having established our claim, we can bound

$$\begin{aligned} \mu_k \beta_k \left\| \frac{z^\mu \bar{z}^\nu}{z_k} \frac{z^{\alpha+\alpha'} \bar{z}^{\beta+\beta'}}{\bar{z}_k} \right\|_{L_t^1} &\leq \mu_k \beta_k \left\| \frac{z^\mu \bar{z}^\nu}{z_k \bar{z}_\ell} \right\|_{L_t^\infty} \left\| \bar{z}_\ell \frac{z^\alpha \bar{z}^\beta}{\bar{z}_k} \right\|_{L_t^2} \|z^{\alpha'} \bar{z}^{\beta'}\|_{L_t^2} \\ &\leq C(C_2, C_3) \epsilon^4 \end{aligned} \quad (4.47)$$

by the fact that the monomials in  $Z_0$  have degree at least 4 (and so  $\| \frac{z^\mu \bar{z}^\nu}{z_k \bar{z}_\ell} \|_{L_t^\infty} \leq C(C_3) \epsilon^2$ ) and that the monomial  $\bar{z}_\ell \frac{z^\alpha \bar{z}^\beta}{\bar{z}_k}$  belongs to the class of monomials with indexes in  $M \cup M'$ . Same argument and bounds hold for the second summation in (4.42). This yields (4.41).  $\square$

**Lemma 4.11.** *There is a fixed  $\epsilon_0 > 0$  such that assuming (4.4)–(4.6) and for  $\epsilon \in (0, \epsilon_0)$  we have*

$$\| \text{second} + \text{third line rhs}(4.31) \|_{L_t^1[0, T]} \leq C(C_2, C_3) \epsilon^3. \quad (4.48)$$

*Proof.* We will only bound

$$\mu_k \nu_j \| e^{i(m+m')t} \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k \bar{z}_j} \text{rhs}(4.12)_k \bar{\zeta}_j \|_{L_t^1} \leq C(C_2, C_3) \epsilon^3, \quad (4.49)$$

with terms from the first line in rhs(4.16). In particular we assume  $(m, \mu, \nu) \in M$  and  $(m', \mu', \nu') \in M'$ . The other terms can be treated similarly. To begin with, we will show

$$\mu_k \nu_j \| e^{i(m+m')t} \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k \bar{z}_j} \bar{\zeta}_j \partial_{\bar{z}_k} Z_0(t, z) \|_{L_t^1} \leq C(C_2, C_3) \epsilon^4. \quad (4.50)$$

It will be enough to consider

$$\mu_k \nu_j \nu_k'' \left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k \bar{z}_j} \frac{z^{\mu''} \bar{z}^{\nu''}}{\bar{z}_k} \bar{\zeta}_j \right\|_{L_t^1} \leq C(C_2, C_3) \epsilon^4, \quad (4.51)$$

with  $(\mu'', \nu'')$  as in (3.18). By the argument before (4.46) we can conclude that (3.18) and hypothesis (H8) imply that there is at least one index  $\mu''_\ell \neq 0$  such that  $\lambda_\ell = \lambda_k$ . To prove (4.51) we substitute in the rhs  $\bar{\zeta}_j = \bar{z}_j + (\bar{\zeta}_j - \bar{z}_j)$ . Then

$$\begin{aligned} &\left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k \bar{z}_j} \frac{z^{\mu''} \bar{z}^{\nu''}}{\bar{z}_k} (\bar{\zeta}_j - \bar{z}_j) \right\|_{L_t^1} \leq \\ &\left\| \frac{z^{\mu''} \bar{z}^{\nu''}}{z_\ell \bar{z}_k} \right\|_{L_t^\infty} \left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k \bar{z}_j} z_\ell \right\|_{L_t^2} \|\zeta - z\|_{L_t^2} \leq C(C_2, C_3) \epsilon^5, \end{aligned} \quad (4.52)$$

where we have used that the first and the last factors in the second line of (4.52) contribute at least  $\epsilon^2$ , and the middle one  $\epsilon$ . To complete (4.51) we prove

$$\mu_k \nu_j \nu_k'' \left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k} \frac{z^{\mu''} \bar{z}^{\nu''}}{\bar{z}_k} \right\|_{L_t^1} \leq C(C_2, C_3) \epsilon^4. \quad (4.53)$$



We have by (4.5)–(4.6)

$$\begin{aligned} \left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'}}{z_k} \frac{z^{\mu''} \bar{z}^{\nu''}}{\bar{z}_k} \right\|_{L_t^1} &\leq \left\| \frac{z^{\mu''} \bar{z}^{\nu''}}{z_\ell \bar{z}_k} \right\|_{L_t^\infty} \left\| \frac{z^{\mu+\mu'} \bar{z}^{\nu+\nu'} z_\ell}{z_k} \right\|_{L_t^1} \leq \\ C_3^2 \epsilon^2 \left\| \frac{z^\mu z_\ell}{z_k} \bar{z}^\nu \right\|_{L_t^2} \left\| z^{\mu'} \bar{z}^{\nu'} \right\|_{L_t^2} &\leq C_3^2 C_2^2 \epsilon^4. \end{aligned} \quad (4.54)$$

Hence we have proved (4.50). Rewriting  $\text{rhs}(4.12) = \text{rhs}(4.13) + (4.14) + (4.15)$ , to complete the proof of (4.49) it is enough to prove (4.55)–(4.56) below.

We need to show for  $(m, \alpha, \beta)$  either in  $M$  or in  $M'$ , for  $(m', \alpha', \beta')$  in  $M'$  and for  $(\mu, \nu)$  sums of exponents in either of the two lines in (4.16),

$$\mu_k \nu_j \beta_k \left\| \frac{z^{\mu} \bar{z}^{\nu}}{\bar{z}_j z_k} e^{i(m \pm m')t} \frac{z^{\alpha+\alpha'} \bar{z}^{\beta+\beta'}}{\bar{z}_k} \bar{\zeta}_j \right\|_{L_t^1} \leq C(C_2, C_3) \epsilon^4. \quad (4.55)$$

We also need to show:

$$\mu_k \nu_j \left\| \frac{z^{\mu} \bar{z}^{\nu}}{\bar{z}_j z_k} \mathcal{E}_k \bar{\zeta}_j \right\|_{L_t^1} \leq C(C_2, C_3) \epsilon^4. \quad (4.56)$$

Let us start with (4.55). Substituting  $\bar{\zeta}_j = \bar{z}_j + (\bar{\zeta}_j - \bar{z}_j)$  and focusing for definiteness on terms of the first line of (4.16), we reduce to

$$\begin{aligned} \mu_k \nu_j \beta_k \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''} z^{\alpha+\alpha'} \bar{z}^{\beta+\beta'}}{z_k \bar{z}_k} \right\|_{L_t^1} &\leq C(C_2, C_3) \epsilon^4, \\ \mu_k \nu_j \beta_k \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''} z^{\alpha+\alpha'} \bar{z}^{\beta+\beta'}}{\bar{z}_j z_k \bar{z}_k} (z_j - \bar{\zeta}_j) \right\|_{L_t^1} &\leq C(C_2, C_3) \epsilon^4, \end{aligned} \quad (4.57)$$

for some  $(\tilde{m}', \mu', \nu') \in M$  and  $(\tilde{m}'', \mu'', \nu'') \in M'$ , with  $\mu_k$  (resp.  $\nu_k$ ) equal either to  $\mu'_k$  or  $\mu''_k$  (resp.  $\nu'_k$  or  $\nu''_k$ ). Inequalities (4.57) can be easily proved using previous arguments. Finally let us turn now to (4.56). Here too for definiteness we prove, for some  $(\tilde{m}', \mu', \nu') \in M$  and  $(\tilde{m}'', \mu'', \nu'') \in M'$ ,

$$\mu'_k \nu'_j \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{\bar{z}_j z_k} \mathcal{E}_k \bar{\zeta}_j \right\|_{L_t^1} \leq C(C_2, C_3) \epsilon^4. \quad (4.58)$$

We have

$$\mu'_k \nu'_j \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{\bar{z}_j z_k} \mathcal{E}_k \bar{\zeta}_j \right\|_{L_t^1} \leq \mu'_k \nu'_j \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{\bar{z}_j z_k} \bar{\zeta}_j \right\|_{L_t^2} \|\mathcal{E}_k\|_{L_t^2}. \quad (4.59)$$

We substitute  $\bar{\zeta}_j = \bar{z}_j + (\bar{\zeta}_j - \bar{z}_j)$ . Then we have

$$\begin{aligned} \mu'_k \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{z_k} \right\|_{L_t^2} &\leq \mu'_k \left\| \frac{z^{\mu'} \bar{z}^{\nu'}}{z_k} \right\|_{L_t^\infty} \left\| z^{\mu''} \bar{z}^{\nu''} \right\|_{L_t^2} \leq C(C_2, C_3) \epsilon^3, \\ \mu'_k \nu'_j \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{\bar{z}_j z_k} (\bar{\zeta}_j - \bar{z}_j) \right\|_{L_t^2} &\leq \mu'_k \nu'_j \left\| \frac{z^{\mu'+\mu''} \bar{z}^{\nu'+\nu''}}{\bar{z}_j z_k} \right\|_{L_t^\infty} \|\bar{\zeta} - z\|_{L_t^2} \\ &\leq C(C_2, C_3) \epsilon^3. \end{aligned} \quad (4.60)$$

□

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